In general, the substructural scheme pushes the real parts of the eigenvalues farther left in the complex plane.

## Conclusions

The numerical results indicate the overall accuracy and numerical efficiency of the substructural control technique as compared to the LQG design by the complete model. The proposed method in general assumes no restrictions on the types of substructures and on the locations of sensors and actuators.

## Acknowledgment

The first author gratefully acknowledges the support of King Fahd University of Petroleum and Minerals in carrying out this research.

## References

[1]Pan, T. S., "Robust Structural and Control Design for Flexible Structures," Ph.D. Dissertation, Purdue Univ., West Lafayette, IN, Dec. 1989, Chap. 2.

[2]Sunar, M., and Rao, S. S., "Substructure Decomposition Method for the Control Design of Large Flexible Structures," *AIAA Journal*, Vol. 30, No. 10, 1992, pp. 2573–2575.

[3]Su, T.-J., Babuska, V., and Craig, R. R., Jr., "Substructure-Based Controller Design Method for Flexible Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 5, 1995, pp. 1053–1061.

A. M. Waas
*Associate Editor*

# Fracture Mechanics of Mode Separation Based on Beam Theory

R. K. Pandey*
*General Electric Corporate Research and Development, Niskayuna, New York 12309*

## Introduction

LAMINATED structures are finding increasing applications in a wide range of industrial and consumer products such as printed circuit boards, aircrafts, ships, automobiles, sporting goods, etc. Quite often these structures delaminate during service conditions, causing a change in the part performance. Estimation of the part performance using analytical methods is very complex and, hence, is currently evaluated using cumbersome two- or three-dimensional numerical techniques. Unfortunately, these numerical methods are time consuming and computationally very expensive. This has led to a search for a simplistic model that can be used for solving these types of fracture mechanics problems efficiently, thereby reducing the design cost and cycle time. Based on the loading and geometry, a typical delamination is driven by a mix mode fracture. Evaluation of the effect of these delaminations on the structural performance requires separation of the strain energy release rate into its mode I and mode II components.

Suo and Huchinson[1] developed an analytical model to calculate the strain energy release rate for a crack at the interface of two homogeneous layers. Parameters used for separating the strain energy release rate into mode I and II strain energy release rates are obtained through numerical analysis. A crack in a single homogeneous structure is a special case of the model. Schapery and Davidson[2] used classical plate theory for predicting the strain energy release rate and for separating it into its constituents for a cracked plate. However, all of the four parameters needed for the mode separation could not be obtained from the plate theory and the determination of at least one of the parameters required numerical analysis.

Williams[3] proposed a mode separation method using the assumption that the mode II fracture causes identical curvatures in two split segments of a double cantilever beam. However, this assumption provides satisfactory results only for a few trivial cases as discussed in detail by Sun and Pandey.[4] Suo[5] and Nilsson and Storakers[6] also separately proposed new methods for mode separation using classical theories. In most of the cases discussed here, parameters used for the mode separation are either unknown or known only for a few special cases.

Separation of a strain energy release rate into its modes, especially those lying at the interfaces, is complex. In the present work, a method for separating the strain energy release rate for a homogeneous isotropic cracked beamlike structure, as shown in Fig. 1, is studied. The proposed method assumes that equal transverse displacement of the split segments near the crack tip causes mode II fracture and applies analytically obtained crack-tip compliance[4] for separating the modes.

## Problem Statement

Consider a cracked homogeneous isotropic beam shown in Fig. 1. The beam has a crack of length $a$, the thicknesses of the upper and the lower segments of the beam are $h_t$ and $h_b$, and the moments acting on these segments are $M_t$ and $M_b$, respectively. Under the action of the moments just defined, assume that the beam experiences mixed mode fracture, that is, the strain energy release rate has both mode I and mode II components. The total strain energy release rate for the described structure under pure moment loading can be accurately expressed as[4]

$$G = \frac{M_t^2}{2EI_t} + \frac{M_b^2}{2EI_b} - \frac{(M_t + M_b)^2}{2EI_0} \tag{1}$$

where $E$ is the elastic modulus of the material and $I$ is the moment of inertia of different segments of the beam identified by their subscripts; subscript 0 stands for unsplit segment. In the present work, a method is proposed to separate $G$ into its mode I and II components ($G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ ) using the elasticity solution of the near tip compliance and the definition of the virtual crack closure method of strain energy release rate calculation.

## Solution Approach

Consider the beam shown in Fig. 1a under a general set of moment loads. The equivalent moments on the near tip cross section are $M_t$ and $M_b$, which can be considered to be made of two sets of moments as shown in Fig. 1b, one causing mode I fracture and the other causing mode II fracture. Let $M_{\mathrm{I}}$ and $\alpha M_{\mathrm{I}}$ be the moments

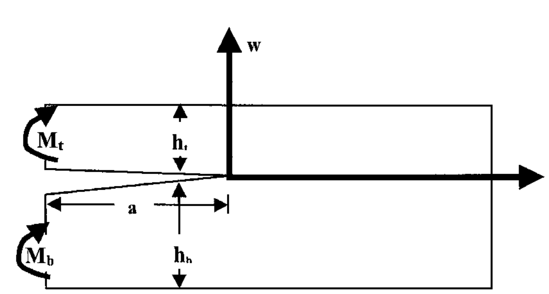


**Fig. 1a Cracked beam under moment loading.**

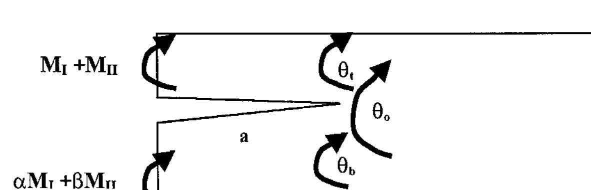


**Fig. 1b Equivalent mode I and mode II moment components.**

Received 16 January 2000; revision received 10 July 2000; accepted for publication 19 July 2000. 

*Senior Staff Engineer, 1 Research Circle; currently Team Leader, TECO–Westinghouse Motor Company, 5100 N. IH 35, Round Rock, TX 78681; pandeyr@teco-wmc.com.

causing mode I fracture and $M_{\mathrm{II}}$ and $\beta M_{\mathrm{II}}$ be the moments causing mode II fracture, that is,

$$M_t = M_{\mathrm{I}} + M_{\mathrm{II}} \tag{2}$$

$$M_b = \alpha M_{\mathrm{I}} + \beta M_{\mathrm{II}} \tag{3}$$

where $\alpha$ and $\beta$ are parameters used for separating the moments into two parts relating to the two fracture modes. Strain energy release rates for the cracked beam for the mode I and mode II cases can be written as

$$G_{\mathrm{I}} = \frac{M_{\mathrm{I}}^2}{2EI_t} + \frac{\alpha^2 M_{\mathrm{I}}^2}{2EI_b} - \frac{(1+\alpha)^2 M_{\mathrm{I}}^2}{2EI_0} \tag{4}$$

$$G_{\mathrm{II}} = \frac{M_{\mathrm{II}}^2}{2EI_t} + \frac{\beta^2 M_{\mathrm{II}}^2}{2EI_b} - \frac{(1+\beta)^2 M_{\mathrm{II}}^2}{2EI_0} \tag{5}$$

but

$$G = G_{\mathrm{I}} + G_{\mathrm{II}} \tag{6}$$

Substituting $G$ from Eq. (1), $G_{\mathrm{I}}$ and $G_{\mathrm{II}}$ from Eqs. (4) and (5), and $M_t$ and $M_b$ from Eqs. (2) and (3) into Eq. (6) and comparing both sides of Eq. (6), the following relationship for mode separation parameters $\alpha$ and $\beta$ is obtained:

$$\alpha + \beta + \alpha\beta(1 - I_0/I_b) + (1 - I_0/I_t) = 0 \tag{7}$$

A second relationship for the mode separation parameter is obtained from the definition of virtual crack closure techniques.

Deflection of the upper and the lower segments of the beam relative to the crack tip can be obtained based on the simple beam theory,

$$w_t = (M_{\mathrm{II}}/2EI_t)x^2 - (\theta_t - \theta_0)x \tag{8}$$

$$w_b = (\beta M_{\mathrm{II}}/2EI_b)x^2 - (\theta_b - \theta_0)x \tag{9}$$

where $x < 0$; $\theta_t$ and $\theta_b$ are rotations of the cross sections of the upper and lower beam segments, respectively, at the crack tip; and $\theta_0$ is the rotation of the uncracked portion of the beam near the crack tip. However, the strain energy release rate depends on the stress and the displacement field local to the crack tip and is defined using the virtual crack closure technique as

$$G = \lim_{\Delta a \to 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{yj}(\Delta a - x, 0) u_j(x, 0)\, \mathrm{d}x \tag{10}$$

where $G = G_{\mathrm{I}}$ for $j = y$ and $G = G_{\mathrm{II}}$ for $j = x$. For $0 < x < \Delta a$ and limit $\Delta a \to 0$, quadratic terms in $x$ in Eqs. (8) and (9) approach zero faster than the linear terms. Based on these observations and that the strain energy release rate is defined using the near tip deformations and stresses, Eqs. (8) and (9) can be reduced to describe the near-tip displacements as

$$w_t = (\theta_t - \theta_0)x \tag{11}$$

$$w_b = (\theta_b - \theta_0)x \tag{12}$$

A mode II fracture will take place only if there is no relative transverse displacement between the top and the bottom segments of the beam. This is possible only if the top and the bottom segments have the same near-tip transverse displacement. Hence, for mode II fracture, Eqs. (11) and (12) can be reduced to

$$\theta_t - \theta_0 = \theta_b - \theta_0 \tag{13}$$

Sun and Pandey[4] have shown that the relative rotations of the upper and the lower beam segments can be related to the near-tip moments in the beam segments by

$$\begin{Bmatrix} \theta_t - \theta_0 \\ \theta_b - \theta_0 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} M_t \\ M_b \end{bmatrix} \tag{14}$$

where $[S]$ is a function of the geometry of the split segments. Substituting $M_{\mathrm{II}}$ for $M_t$ and $\beta M_{\mathrm{II}}$ for $M_b$ in Eq. (14) and using the condition described in Eq. (13) for mode II fracture, $\beta$ can be expressed as

$$\beta = \frac{S_{11} - S_{21}}{S_{22} - S_{12}} \tag{15}$$

When $\beta$ is substituted in Eq. (8), $\alpha$ is determined. These values of mode separation parameters, $\alpha$ and $\beta$, can be substituted in Eqs. (4) and (5) to obtain the mode I and II strain energy release rates for the cracked plate/beam.

## Summary

A method is proposed for separating the strain energy release rate into its constituent modes using analytical methods for a cracked beam/platelike structure. The method uses a near-tip load–displacement relationship and virtual crack closure technique to separate the mode. The accuracy of the mode separation is dependent on the accuracy of the near-tip compliance, matrix $[S]$ of the cross-sectional plane, that is, the load–displacement relationship in the near crack-tip area. The strain energy release rate calculated using the near-tip compliance tends to overpredict the strain energy release rate for mode II loading and underpredict for mode I loading.[4] Thus, the variation in total strain energy release rate, due to the variation in the prediction of the calculated compliance,[4] is reduced due to the aforementioned complementary effects, hence, the error in total value of the strain energy release rate as compared to that in its individual components.

## Acknowledgments

The author would like to thank C. T. Sun, Purdue University, for continuously being the best advisor and Teco–Westinghouse Motor Company for support and encouragement.

## References

[1] Suo, Z., and Huchinson, J. H., "Interface Crack Between Two Elastic Layers," *International Journal of Fracture*, Vol. 43, 1990, pp. 1–18.

[2] Schapery, R. A., and Davidson, B. D., "Prediction of Energy Release Rate for Mixed-Mode Delamination Using Classical Plate Theory," *Applied Mechanics Review*, Vol. 43, No. 5, Pt. 2, 1990, pp. S281–S287.

[3] Williams, J. G., "On the Calculation of Strain Energy Release Rate for Cracked Laminates," *International Journal of Fracture*, Vol. 36, 1988, pp. 101–109.

[4] Sun, C. T., and Pandey, R. K., "Improved Method for Calculating Strain Energy Release Rate Based on Beam Theory," *AIAA Journal*, Vol. 32, No. 1, 1994, pp. 184–189.

[5] Suo, Z., "Delamination Specimens for Orthotropic Materials," *Journal of Applied Mechanics*, Vol. 57, 1990, pp. 627–634.

[6] Nilsson, K.-F., and Storakers, B., "On Interface Crack Growth in Composite Plates," *Transactions of the ASME*, Vol. 59, 1992, pp. 530–538.

E. R. Johnson
*Associate Editor*